Integral Calculus - Exercises

6.1 Antidifferentiation. The Indefinite Integral

In problems 1 through 7, find the indicated integral.

1. $\int \sqrt{x}dx$

   Solution.

$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C = \frac{2}{3}x\sqrt{x} + C.$$

2. $\int 3e^x dx$

   Solution.

$$\int 3e^x dx = 3\int e^x dx = 3e^x + C.$$

3. $\int (3x^2 - \sqrt{5x} + 2)dx$

   Solution.

$$\begin{aligned} \int (3x^2 - \sqrt{5x} + 2)dx &= 3\int x^2 dx - \sqrt{5}\int \sqrt{x}dx + 2\int dx = \\ &= 3 \cdot \frac{1}{3}x^3 - \sqrt{5} \cdot \frac{2}{3}x\sqrt{x} + 2x + C = \\ &= x^3 - \frac{2}{3}x\sqrt{5x} + 2x + C. \end{aligned}$$

4. $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right) dx$

   Solution.

$$\begin{aligned} \int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right) dx &= \frac{1}{2}\int \frac{1}{x}dx - 2\int x^{-2}dx + 3\int x^{-\frac{1}{2}}dx = \\ &= \frac{1}{2}\ln|x| - 2 \cdot (-1)x^{-1} + 3 \cdot 2x^{\frac{1}{2}} + C = \\ &= \frac{\ln|x|}{2} + \frac{2}{x} + 6\sqrt{x} + C. \end{aligned}$$

5. $\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx$
Solution.

$$\begin{aligned}\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx &= 2\int e^x dx + 6\int \frac{1}{x} dx + \ln 2 \int dx = \\ &= 2e^x + 6\ln|x| + (\ln 2)x + C.\end{aligned}$$

6. $\int \frac{x^2+3x-2}{\sqrt{x}} dx$
Solution.

$$\begin{aligned}\int \frac{x^2+3x-2}{\sqrt{x}} dx &= \int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx - 2\int x^{-\frac{1}{2}} dx = \\ &= \frac{2}{5}x^{\frac{5}{2}} + 3\cdot\frac{2}{3}x^{\frac{3}{2}} - 2\cdot 2x^{\frac{1}{2}} + C = \\ &= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C = \\ &= \frac{2}{5}x^2\sqrt{x} + 2x\sqrt{x} - 4\sqrt{x} + C.\end{aligned}$$

7. $\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx$
Solution.

$$\begin{aligned}\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx &= \int (x^2 - 5x^3 - 2x + 10x^2) dx = \\ &= \int (-5x^3 + 11x^2 - 2x) dx = \\ &= -5\cdot\frac{1}{4}x^4 + 11\cdot\frac{1}{3}x^3 - 2\cdot\frac{1}{2}x^2 + C = \\ &= -\frac{5}{4}x^4 + \frac{11}{3}x^3 - x^2 + C.\end{aligned}$$

8. Find the function f whose tangent has slope $x^3 - \frac{2}{x^2} + 2$ for each value of x and whose graph passes through the point $(1, 3)$.
Solution. The slope of the tangent is the derivative of f. Thus

$$f'(x) = x^3 - \frac{2}{x^2} + 2$$

and so $f(x)$ is the indefinite integral

$$\begin{aligned}f(x) &= \int f'(x) dx = \int \left(x^3 - \frac{2}{x^2} + 2\right) dx = \\ &= \frac{1}{4}x^4 + \frac{2}{x} + 2x + C.\end{aligned}$$

Using the fact that the graph of f passes through the point $(1,3)$ you get

$$3 = \frac{1}{4} + 2 + 2 + C \quad \text{or} \quad C = -\frac{5}{4}.$$

Therefore, the desired function is $f(x) = \frac{1}{4}x^4 + \frac{2}{x} + 2x - \frac{5}{4}$.

9. It is estimated that t years from now the population of a certain lakeside community will be changing at the rate of $0.6t^2 + 0.2t + 0.5$ thousand people per year. Environmentalists have found that the level of pollution in the lake increases at the rate of approximately 5 units per 1000 people. By how much will the pollution in the lake increase during the next 2 years?

   Solution. Let $P(t)$ denote the population of the community t years from now. Then the rate of change of the population with respect to time is the derivative

$$\frac{dP}{dt} = P'(t) = 0.6t^2 + 0.2t + 0.5.$$

   It follows that the population function $P(t)$ is an antiderivative of $0.6t^2 + 0.2t + 0.5$. That is,

$$\begin{aligned} P(t) &= \int P'(t)dt = \int (0.6t^2 + 0.2t + 0.5)dt = \\ &= 0.2t^3 + 0.1t^2 + 0.5t + C \end{aligned}$$

   for some constant C. During the next 2 years, the population will grow on behalf of

$$\begin{aligned} P(2) - P(0) &= 0.2 \cdot 2^3 + 0.1 \cdot 2^2 + 0.5 \cdot 2 + C - C = \\ &= 1.6 + 0.4 + 1 = 3 \text{ thousand people.} \end{aligned}$$

   Hence, the pollution in the lake will increase on behalf of $5 \cdot 3 = 15$ units.

10. An object is moving so that its speed after t minutes is $v(t) = 1+4t+3t^2$ meters per minute. How far does the object travel during 3rd minute?

    Solution. Let $s(t)$ denote the displacement of the car after t minutes. Since $v(t) = \frac{ds}{dt} = s'(t)$ it follows that

$$s(t) = \int v(t)dt = \int (1 + 4t + 3t^2)dt = t + 2t^2 + t^3 + C.$$

    During the 3rd minute, the object travels

$$\begin{aligned} s(3) - s(2) &= 3 + 2 \cdot 9 + 27 + C - 2 - 2 \cdot 4 - 8 - C = \\ &= 30 \text{ meters.} \end{aligned}$$

Homework

In problems 1 through 13, find the indicated integral. Check your answers by differentiation.

1.	$\int x^5 dx$	2.	$\int x^{\frac{3}{4}} dx$
3.	$\int \frac{1}{x^2} dx$	4.	$\int 5dx$
5.	$\int (x^{\frac{1}{2}} - 3x^{\frac{2}{3}} + 6)dx$	6.	$\int \left(3\sqrt{x} - \frac{2}{x^3} + \frac{1}{x}\right) dx$
7.	$\int \left(\frac{e^x}{2} + x\sqrt{x}\right) dx$	8.	$\int \left(\sqrt{x^3} - \frac{1}{2\sqrt{x}} + \sqrt{2}\right) dx$
9.	$\int \left(\frac{1}{3x} - \frac{3}{2x^2} + e^2 + \frac{\sqrt{x}}{2}\right) dx$	10.	$\int \frac{x^2+2x+1}{x^2} dx$
11.	$\int x^3 \left(2x + \frac{1}{x}\right) dx$	12.	$\int \sqrt{x}(x^2 - 1)dx$
13.	$\int x(2x + 1)^2 dx$		

14. Find the function whose tangent has slope $4x + 1$ for each value of x and whose graph passes through the point $(1, 2)$.

15. Find the function whose tangent has slope $3x^2 + 6x - 2$ for each value of x and whose graph passes through the point $(0, 6)$.

16. Find a function whose graph has a relative minimum when $x = 1$ and a relative maximum when $x = 4$.

17. It is estimated that t months from now the population of a certain town will be changing at the rate of $4 + 5t^{\frac{2}{3}}$ people per month. If the current population is 10000, what will the population be 8 months from now?

18. An environmental study of a certain community suggests that t years from now the level of carbon monoxide in the air will be changing at the rate of $0.1t + 0.1$ parts per million per year. If the current level of carbon monoxide in the air is 3.4 parts per million, what will the level be 3 years from now?

19. After its brakes are applied, a certain car decelerates at the constant rate of 6 meters per second per second. If the car is traveling at 108 kilometers per hour when the brakes are applied, how far does it travel before coming to a complete stop? (Note: 108 kmph is the same as 30 mps.)

20. Suppose a certain car supplies a constant deceleration of A meters per second per second. If it is traveling at 90 kilometers per hour (25 meters per second) when the brakes are applied, its stopping distance is 50 meters.

    (a) What is A?

(b) What would the stopping distance have been if the car had been traveling at only 54 kilometers per hour when the brakes were applied?

(c) At what speed is the car traveling when the brakes are applied if the stopping distance is 56 meters?

Results.

1.	$\frac{1}{6}x^6 + C$	2.	$\frac{4}{7}x^{\frac{7}{4}} + C$
3.	$-\frac{1}{x} + C$	4.	$5x + C$
5.	$\frac{2}{3}x^{\frac{3}{2}} - \frac{9}{5}x^{\frac{5}{3}} + 6x + C$	6.	$2x^{\frac{3}{2}} + \frac{1}{x^2} + \ln\|x\| + C$
7.	$\frac{1}{2}e^x + \frac{2}{5}x^{\frac{5}{2}} + C$	8.	$\frac{2}{5}\sqrt{(x^3)}x - \sqrt{x} + \sqrt{2}x + C$
9.	$\frac{1}{3}\ln\|x\| + \frac{3}{2x} + e^2x + \frac{1}{3}x^{\frac{3}{2}} + C$	10.	$x - \frac{1}{x} + 2\ln x + C$
11.	$\frac{2}{5}x^5 + \frac{1}{3}x^3 + C$	12.	$\frac{2}{7}x^{\frac{7}{2}} - \frac{2}{3}x^{\frac{3}{2}} + C$
13.	$x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 + C$		

14. $f(x) = 2x^2 + x - 1$

15. $f(x) = x^3 + 3x^2 - 2x + 6$

16. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$; not unique

17. 10128

18. 4.15 parts per million

19. 75 meters

20. (a) $A = 6.25$

(b) 42 meters

(c) 120.37 kilometers per hour

6.2 Integration by Substitution

In problems 1 through 8, find the indicated integral.

1. $\int (2x+6)^5 dx$
Solution. Substituting $u = 2x+6$ and $\frac{1}{2}du = dx$, you get

$$\int (2x+6)^5 dx = \frac{1}{2}\int u^5 du = \frac{1}{12}u^6 + C = \frac{1}{12}(2x+6)^6 + C.$$

2. $\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx$
Solution. Substituting $u = x-1$ and $du = dx$, you get

$$\begin{aligned}\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx &= \int (u^5 + 3u^2 + 5)du = \\ &= \frac{1}{6}u^6 + u^3 + 5u + C = \\ &= \frac{1}{6}(x-1)^6 + (x-1)^3 + 5(x-1) + C.\end{aligned}$$

Since, for a constant C, $C-5$ is again a constant, you can write

$$\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx = \frac{1}{6}(x-1)^6 + (x-1)^3 + 5x + C.$$

3. $\int xe^{x^2} dx$
Solution. Substituting $u = x^2$ and $\frac{1}{2}du = xdx$, you get

$$\int xe^{x^2} dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

4. $\int x^5 e^{1-x^6} dx$
Solution. Substituting $u = 1-x^6$ and $-\frac{1}{6}du = x^5 dx$, you get

$$\int x^5 e^{1-x^6} dx = -\frac{1}{6}\int e^u du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{1-x^6} + C.$$

5. $\int \frac{2x^4}{x^5+1} dx$
Solution. Substituting $u = x^5+1$ and $\frac{2}{5}du = 2x^4 dx$, you get

$$\int \frac{2x^4}{x^5+1} dx = \frac{2}{5}\int \frac{1}{u} du = \frac{2}{5}\ln|u| + C = \frac{2}{5}\ln\left|x^5+1\right| + C.$$

6. $\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}dx$
Solution. Substituting $u = x^4 - x^2 + 6$ and $\frac{5}{2}du = (10x^3 - 5x)dx$, you get

$$\begin{aligned}\int \frac{10x^3 - 5x}{\sqrt{x^4 - x^2 + 6}}dx &= \frac{5}{2}\int \frac{1}{\sqrt{u}}du = \frac{5}{2}\int u^{-\frac{1}{2}}du = \frac{5}{2} \cdot 2u^{\frac{1}{2}} + C = \\ &= 5\sqrt{x^4 - x^2 + 6} + C.\end{aligned}$$

7. $\int \frac{1}{x \ln x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{1}{x \ln x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C.$$

8. $\int \frac{\ln x^2}{x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{\ln x^2}{x}dx = \int \frac{2\ln x}{x}dx = 2\int u du = 2 \cdot \frac{1}{2}u^2 + C = (\ln x)^2 + C.$$

9. Use an appropriate change of variables to find the integral

$$\int (x+1)(x-2)^9 dx.$$

Solution. Substituting $u = x - 2$, $u + 3 = x + 1$ and $du = dx$, you get

$$\begin{aligned}\int (x+1)(x-2)^9 dx &= \int (u+3)u^9 du = \int (u^{10} + 3u^9)du = \\ &= \frac{1}{11}u^{11} + \frac{3}{10}u^{10} + C = \\ &= \frac{1}{11}(x-2)^{11} + \frac{3}{10}(x-2)^{10} + C.\end{aligned}$$

10. Use an appropriate change of variables to find the integral

$$\int (2x+3)\sqrt{2x-1}dx.$$

Solution. Substituting $u = 2x - 1$, $u + 4 = 2x + 3$ and $\frac{1}{2}du = dx$, you get

$$\begin{aligned}
\int (2x+3)\sqrt{2x-1}dx &= \frac{1}{2}\int (u+4)\sqrt{u}du = \frac{1}{2}\int u^{\frac{3}{2}}du + 2\int u^{\frac{1}{2}}du = \\
&= \frac{1}{2}\cdot\frac{2}{5}u^{\frac{5}{2}} + 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C = \\
&= \frac{1}{5}(2x-1)^{\frac{5}{2}} + \frac{4}{3}(2x-1)^{\frac{3}{2}} + C = \\
&= \frac{1}{5}(2x-1)^2\sqrt{2x-1} + \frac{4}{3}(2x-1)\sqrt{2x-1} + C = \\
&= (2x-1)\sqrt{2x-1}\left(\frac{2}{5}x - \frac{1}{5} + \frac{4}{3}\right) + C = \\
&= \left(\frac{2}{5}x + \frac{17}{25}\right)(2x-1)\sqrt{2x-1} + C.
\end{aligned}$$

Homework

In problems 1 through 18, find the indicated integral and check your answer by differentiation.

1.	$\int e^{5x}dx$	2.	$\int \sqrt{4x-1}dx$
3.	$\int \frac{1}{3x+5}dx$	4.	$\int e^{1-x}dx$
5.	$\int 2xe^{x^2-1}dx$	6.	$\int x(x^2+1)^5dx$
7.	$\int 3x\sqrt{x^2+8}dx$	8.	$\int x^2(x^3+1)^{\frac{3}{4}}dx$
9.	$\int \frac{x^2}{(x^3+5)^2}dx$	10.	$\int (x+1)(x^2+2x+5)^{12}dx$
11.	$\int (3x^2-1)e^{x^3-x}dx$	12.	$\int \frac{3x^4+12x^3+6}{x^5+5x^4+10x+12}dx$
13.	$\int \frac{3x-3}{(x^2-2x+6)^2}dx$	14.	$\int \frac{6x-3}{4x^2-4x+1}dx$
15.	$\int \frac{\ln 5x}{x}dx$	16.	$\int \frac{1}{x(\ln x)^2}dx$
17.	$\int \frac{2x\ln(x^2+1)}{x^2+1}dx$	18.	$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

In problems 19 through 23, use an appropriate change of variables to find the indicated integral.

19.	$\int \frac{x}{x-1}dx$	20.	$\int x\sqrt{x+1}dx$
21.	$\int \frac{x}{(x-5)^6}dx$	22.	$\int \frac{x+3}{(x-4)^2}dx$
23.	$\int \frac{x}{2x+1}dx$		

24. Find the function whose tangent has slope $x\sqrt{x^2+5}$ for each value of x and whose graph passes through the point $(2, 10)$.

25. Find the function whose tangent has slope $\frac{2x}{1-3x^2}$ for each value of x and whose graph passes through the point $(0, 5)$.

26. A tree has been transplanted and after x years is growing at the rate of $1 + \frac{1}{(x+1)^2}$ meters per year. After two years it has reached a height of five meters. How tall was it when it was transplanted?

27. It is projected that t years from now the population of a certain country will be changing at the rate of $e^{0.02t}$ million per year. If the current population is 50 million, what will the population be 10 years from now?

Results.

1. $\frac{1}{5}e^{5x} + C$
2. $\frac{1}{6}(4x-1)\sqrt{4x-1} + C$
3. $\frac{1}{3}\ln|3x+5| + C$
4. $-e^{1-x} + C$
5. $e^{x^2-1} + C$
6. $\frac{1}{12}(x^2+1)^6 + C$
7. $(x^2+8)\sqrt{x^2+8} + C$
8. $\frac{4}{21}(x^3+1)^{\frac{7}{4}} + C$
9. $-\frac{1}{3(x^3+5)} + C$
10. $\frac{1}{26}(x^2+2x+5)^{13} + C$
11. $e^{x^3-x} + C$
12. $\frac{3}{5}\ln|x^5+5x^4+10x+12| + C$
13. $-\frac{3}{2(x^2-2x+6)} + C$
14. $\frac{3}{2}\ln|2x-1| + C$
15. $\frac{1}{2}\ln^2 5x + C$
16. $-\frac{1}{\ln x} + C$
17. $\frac{1}{2}\ln^2(x^2+1) + C$
18. $2e^{\sqrt{x}} + C$
19. $x + \ln|x-1| + C$
20. $\frac{2}{5}(x+1)^2\sqrt{x+1} - \frac{2}{3}(x+1)\sqrt{x+1} + C$
21. $-\frac{1}{(x-5)^5} - \frac{1}{4(x-5)^4} + C$
22. $-\frac{7}{x-4} + \ln|x-4| + C$
23. $\frac{2x+1}{4}x - \frac{1}{4}\ln|2x+1| + C$

24. $f(x) = \frac{1}{3}(x^2+5)\sqrt{x^2+5} + 1$

25. $f(x) = -\frac{1}{3}\ln|1-3x^2| + 5$

26. $\frac{7}{3}$ meters

27. 61 million

6.3 Integration by Parts

In problems 1 through 9, use integration by parts to find the given integral.

1. $\int xe^{0.1x}dx$
Solution. Since the factor $e^{0.1x}$ is easy to integrate and the factor x is simplified by differentiation, try integration by parts with

$$g(x) = e^{0.1x} \quad \text{and} \quad f(x) = x.$$

Then,

$$G(x) = \int e^{0.1x}dx = 10e^{0.1x} \quad \text{and} \quad f'(x) = 1$$

and so

$$\begin{aligned}\int xe^{0.1x}dx &= 10xe^{0.1x} - 10\int e^{0.1x}dx = 10xe^{0.1x} - 100e^{0.1x} + C = \\ &= 10(x-10)e^{0.1x} + C.\end{aligned}$$

2. $\int(3-2x)e^{-x}dx$
Solution. Since the factor e^{-x} is easy to integrate and the factor $3-2x$ is simplified by differentiation, try integration by parts with

$$g(x) = e^{-x} \quad \text{and} \quad f(x) = 3-2x.$$

Then,

$$G(x) = \int e^{-x}dx = -e^{-x} \quad \text{and} \quad f'(x) = -2$$

and so

$$\begin{aligned}\int(3-2x)e^{-x}dx &= (3-2x)(-e^{-x}) - 2\int e^{-x}dx = \\ &= (2x-3)e^{-x} + 2e^{-x} + C = (2x-1)e^{-x} + C.\end{aligned}$$

3. $\int x\ln x^2 dx$
Solution. In this case, the factor x is easy to integrate, while the factor $\ln x^2$ is simplified by differentiation. This suggests that you try integration by parts with

$$g(x) = x \quad \text{and} \quad f(x) = \ln x^2.$$

Then,

$$G(x) = \int xdx = \frac{1}{2}x^2 \quad \text{and} \quad f'(x) = \frac{1}{x^2}2x = \frac{2}{x}$$

and so

$$\begin{aligned}\int x\ln x^2 dx &= \frac{1}{2}x^2\ln x^2 - \int \frac{1}{2}x^2\frac{2}{x}dx = \frac{1}{2}x^2\ln x^2 - \int x dx = \\ &= \frac{1}{2}x^2\ln x^2 - \frac{1}{2}x^2 + C = \frac{1}{2}x^2\left(\ln x^2 - 1\right) + C.\end{aligned}$$

4. $\int x\sqrt{1-x}dx$

Solution. Since the factor $\sqrt{1-x}$ is easy to integrate and the factor x is simplified by differentiation, try integration by parts with

$$g(x) = \sqrt{1-x} \qquad \text{and} \qquad f(x) = x.$$

Then,

$$G(x) = \int \sqrt{1-x}dx = -\frac{2}{3}(1-x)^{\frac{3}{2}} \qquad \text{and} \qquad f'(x) = 1$$

and so

$$\begin{aligned}\int x\sqrt{1-x}dx &= -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\int (1-x)^{\frac{3}{2}}dx = \\ &= -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\left(-\frac{2}{5}(1-x)^{\frac{5}{2}}\right) + C = \\ &= -\frac{2}{3}x(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + C = \\ &= -\frac{2}{3}x(1-x)\sqrt{1-x} - \frac{4}{15}(1-x)^2\sqrt{1-x} + C.\end{aligned}$$

5. $\int (x+1)(x+2)^6 dx$

Solution. Since the factor $(x+2)^6$ is easy to integrate and the factor $x+1$ is simplified by differentiation, try integration by parts with

$$g(x) = (x+2)^6 \qquad \text{and} \qquad f(x) = x+1.$$

Then,

$$G(x) = \int (x+2)^6 dx = \frac{1}{7}(x+2)^7 \qquad \text{and} \qquad f'(x) = 1$$

and so

$$\begin{aligned}\int (x+1)(x+2)^6 dx &= \frac{1}{7}(x+1)(x+2)^7 - \frac{1}{7}\int (x+2)^7 dx = \\ &= \frac{1}{7}(x+1)(x+2)^7 - \frac{1}{7}\frac{1}{8}(x+2)^8 + C = \\ &= \frac{1}{56}\left[8(x+1) - (x+2)\right](x+2)^7 + C = \\ &= \frac{1}{56}(7x+6)(x+2)^7 + C.\end{aligned}$$

6. $\int x^3 e^{2x} dx$

Solution. Since the factor e^{2x} is easy to integrate and the factor x^3 is simplified by differentiation, try integration by parts with

$$g(x) = e^{2x} \qquad \text{and} \qquad f(x) = x^3.$$

Then,

$$G(x) = \int e^{2x} dx = \frac{1}{2}e^{2x} \qquad \text{and} \qquad f'(x) = 3x^2$$

and so

$$\int x^3 e^{2x} dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x} dx.$$

To find $\int x^2 e^{2x} dx$, you have to integrate by parts again, but this time with

$$g(x) = e^{2x} \qquad \text{and} \qquad f(x) = x^2.$$

Then,

$$G(x) = \frac{1}{2}e^{2x} \qquad \text{and} \qquad f'(x) = 2x$$

and so

$$\int x^2 e^{2x} dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x} dx.$$

To find $\int x e^{2x} dx$, you have to integrate by parts once again, this time with

$$g(x) = e^{2x} \qquad \text{and} \qquad f(x) = x.$$

Then,

$$G(x) = \frac{1}{2}e^{2x} \qquad \text{and} \qquad f'(x) = 1$$

and so

$$\int x e^{2x} dx = \frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x} dx = \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}.$$

Finally,

$$\begin{aligned} \int x^3 e^{2x} dx &= \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left[\frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\right)\right] + C = \\ &= \left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right)e^{2x} + C. \end{aligned}$$

7. $\int \frac{\ln x}{x^3} dx$
Solution. In this case, the factor $\frac{1}{x^3}$ is easy to integrate, while the factor $\ln x$ is simplified by differentiation. This suggests that you try integration by parts with

$$g(x) = \frac{1}{x^3} \quad \text{and} \quad f(x) = \ln x.$$

Then,

$$G(x) = \int \frac{1}{x^3} dx = -\frac{1}{2} x^{-2} = -\frac{1}{2x^2} \quad \text{and} \quad f'(x) = \frac{1}{x}$$

and so

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= -\frac{\ln x}{2x^2} + \frac{1}{2} \int \frac{1}{x^3} dx = -\frac{\ln x}{2x^2} + \frac{1}{2} \left(-\frac{1}{2x^2} \right) + C = \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C. \end{aligned}$$

8. $\int x^3 e^{x^2} dx$
Solution. First rewrite the integrand as $x^2 \left(x e^{x^2} \right)$, and then integrate by parts with

$$g(x) = x e^{x^2} \quad \text{and} \quad f(x) = x^2.$$

Then, from Exercise 6.2.3 you get

$$G(x) = \int x e^{x^2} dx = \frac{1}{2} e^{x^2} \quad \text{and} \quad f'(x) = 2x$$

and so

$$\begin{aligned} \int x^3 e^{x^2} dx &= \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C = \\ &= \frac{1}{2} (x^2 - 1) e^{x^2} + C. \end{aligned}$$

9. $\int x^3 (x^2 - 1)^{10} dx$
Solution. First rewrite the integrand as $x^2 [x(x^2 - 1)^{10}]$, and then integrate by parts with

$$g(x) = x(x^2 - 1)^{10} \quad \text{and} \quad f(x) = x^2.$$

Then

$$G(x) = \int x(x^2 - 1)^{10} dx \quad \text{and} \quad f'(x) = 2x.$$

Substituting $u = x^2 - 1$ and $\frac{1}{2}du = xdx$, you get

$$G(x) = \int x(x^2-1)^{10}dx = \frac{1}{2}\int u^{10}du = \frac{1}{22}u^{11} = \frac{1}{22}(x^2-1)^{11}.$$

Then

$$\begin{aligned}\int x^3(x^2-1)^{10}dx &= \frac{1}{22}x^2(x^2-1)^{11} - \frac{1}{22}\int 2x(x^2-1)^{11}dx = \\ &= \frac{1}{22}x^2(x^2-1)^{11} - \frac{1}{22}\frac{1}{12}(x^2-1)^{12} + C = \\ &= \frac{1}{22}x^2(x^2-1)^{11} - \frac{1}{264}(x^2-1)^{12} + C.\end{aligned}$$

(a) Use integration by parts to derive the formula

$$\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx.$$

(b) Use the formula in part (a) to find $\int x^3 e^{5x}dx$.

Solution. (a) Since the factor e^{ax} is easy to integrate and the factor x^n is simplified by differentiation, try integration by parts with

$$g(x) = e^{ax} \qquad \text{and} \qquad f(x) = x^n.$$

Then,

$$G(x) = \int e^{ax}dx = \frac{1}{a}e^{ax} \qquad \text{and} \qquad f'(x) = nx^{n-1}$$

and so

$$\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx.$$

(b) Apply the formula in part (a) with $a = 5$ and $n = 3$ to get

$$\int x^3 e^{5x}dx = \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\int x^2 e^{5x}dx.$$

Again, apply the formula in part (a) with $a = 5$ and $n = 2$ to find the new integral

$$\int x^2 e^{5x}dx = \frac{1}{5}x^2 e^{5x} - \frac{2}{5}\int xe^{5x}dx.$$

Once again, apply the formula in part (a) with $a = 5$ and $n = 1$ to get

$$\int xe^{5x}dx = \frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x}dx = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}$$

and so

$$\begin{aligned}\int x^3e^{5x}dx &= \frac{1}{5}x^3e^{5x} - \frac{3}{5}\left[\frac{1}{5}x^2e^{5x} - \frac{2}{5}\left(\frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}\right)\right] + C = \\ &= \frac{1}{5}\left(x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125}\right)e^{5x} + C.\end{aligned}$$

Homework

In problems 1 through 16, use integration by parts to find the given integral.

1.	$\int xe^{-x}dx$	2.	$\int xe^{\frac{x}{2}}dx$
3.	$\int xe^{-\frac{x}{5}}dx$	4.	$\int(1-x)e^{x}dx$
5.	$\int x\ln 2x dx$	6.	$\int x\sqrt{x-6}dx$
7.	$\int x(x+1)^8dx$	8.	$\int \frac{x}{\sqrt{x+2}}dx$
9.	$\int \frac{x}{\sqrt{2x+1}}dx$	10.	$\int x^2e^{-x}dx$
11.	$\int x^2e^{3x}dx$	12.	$\int x^3e^{x}dx$
13.	$\int x^2\ln x dx$	14.	$\int x(\ln x)^2dx$
15.	$\int \frac{\ln x}{x^2}dx$	16.	$\int x^7(x^4+5)^8dx$

17. Find the function whose tangent has slope $(x+1)e^{-x}$ for each value of x and whose graph passes through the point $(1, 5)$.

18. Find the function whose tangent has slope $x\ln\sqrt{x}$ for each value of $x > 0$ and whose graph passes through the point $(2, -3)$.

19. After t seconds, an object is moving at the speed of $te^{-\frac{t}{2}}$ meters per second. Express the distance the object travels as a function of time.

20. It is projected that t years from now the population of a certain city will be changing at the rate of $t\ln\sqrt{t+1}$ thousand people per year. If the current population is 2 million, what will the population be 5 years from now?

Results.

1. $-(x+1)e^{-x}+C$
2. $(2x-4)e^{\frac{1}{2}x}+C$
3. $-5(x+5)e^{-\frac{1}{5}x}+C$
4. $(2-x)e^{x}+C$
5. $\frac{1}{2}x^2(\ln 2x-\frac{1}{2})+C$
6. $\frac{2}{3}x\,(x-6)^{\frac{3}{2}}-\frac{4}{15}\,(x-6)^{\frac{5}{2}}+C$
7. $\frac{1}{9}x(x+1)^9-\frac{1}{90}(x+1)^{10}+C$
8. $2x\,(x+2)^{\frac{1}{2}}-\frac{4}{3}\,(x+2)^{\frac{3}{2}}+C$
9. $x\,(2x+1)^{\frac{1}{2}}-\frac{1}{3}\,(2x+1)^{\frac{3}{2}}+C$
10. $-(x^2+2x+2)e^{-x}+C$
11. $\frac{1}{3}\left(x^2-\frac{2}{3}x+\frac{2}{9}\right)e^{3x}+C$
12. $(x^3-3x^2+6x-6)e^{x}+C$
13. $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$
14. $\frac{1}{2}x^2\left(\ln^2 x-\ln x+\frac{1}{2}\right)+C$
15. $-\frac{1}{x}(\ln x+1)+C$
16. $\frac{1}{36}x^4(x^4+5)^9-\frac{1}{360}(x^4+5)^{10}+C$

17. $f(x)=-(x+2)e^{-x}+\frac{3}{e}+5$

18. $f(x)=\frac{1}{4}x^2\left(\ln x-\frac{1}{2}\right)-\frac{5}{2}-\ln 2$

19.

   1. $s(t)=-2(t+2)e^{-\frac{t}{2}}+4$

20. 2008875

6.4 The use of Integral tables

In Problems 1 through 5, use one of the integration formulas from a table of integrals (see Appendix) to find the given integral.

1. $\int \frac{dx}{\sqrt{x^2+2x-3}}$
Solution. First rewrite the integrand as

$$\frac{1}{\sqrt{x^2+2x-3}} = \frac{1}{\sqrt{(x+1)^2-4}}$$

and then substitute $u = x+1$ and $du = dx$ to get

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+2x-3}} &= \int \frac{dx}{\sqrt{(x+1)^2-4}} = \int \frac{du}{\sqrt{u^2-4}} = \\ &= \ln\left|u+\sqrt{u^2-4}\right| + C = \ln\left|x+1+\sqrt{(x+1)^2-4}\right| + C = \\ &= \ln\left|x+1+\sqrt{x^2+2x-3}\right| + C.\end{aligned}$$

2. $\int \frac{dx}{1-6x-3x^2}$
Solution. First, rewrite the integrand as

$$\frac{1}{1-6x-3x^2} = \frac{1}{1-3(2x+x^2)} = \frac{1}{4-3(x+1)^2} = \frac{1}{3}\frac{1}{\frac{4}{3}-(x+1)^2}$$

and then substitute $u = x+1$ and $du = dx$ to get

$$\begin{aligned}\int \frac{dx}{1-6x-3x^2} &= \frac{1}{3}\int \frac{dx}{\frac{4}{3}-(x+1)^2} = \frac{1}{3}\int \frac{du}{\frac{4}{3}-u^2} = \\ &= \frac{1}{3}\frac{3}{8}\ln\left|\frac{\frac{4}{3}+u}{\frac{4}{3}-u}\right| + C = \frac{1}{8}\ln\left|\frac{4+3u}{4-3u}\right| + C = \\ &= \frac{1}{8}\ln\left|\frac{7+3x}{1-3x}\right| + C.\end{aligned}$$

3. $\int (x^2+1)^{\frac{3}{2}} dx$
Solution. First rewrite the integrand as

$$(x^2+1)^{\frac{3}{2}} = (x^2+1)\sqrt{x^2+1} = x^2\sqrt{x^2+1} + \sqrt{x^2+1}.$$

Aplly appropriate formulas (see Appendix, formulas 9 and 13), to get

$$\int x^2\sqrt{x^2+1}dx = \frac{x}{8}(1+2x^2)\sqrt{x^2+1} - \frac{1}{8}\ln\left|x+\sqrt{x^2+1}\right| + C$$

and

$$\int \sqrt{x^2+1}dx = \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| + C.$$

Combine these results, to conclude that

$$\int (x^2+1)^{\frac{3}{2}}dx = \frac{1}{8}x(2x^2+5)\sqrt{(x^2+1)} + \frac{3}{8}\ln\left|x+\sqrt{x^2+1}\right| + C.$$

4. $\int \frac{dx}{2-3e^{-x}}$
Solution. Aplly appropriate formula (see Appendix, formula 23), to get

$$\begin{aligned}\int \frac{dx}{2-3e^{-x}} &= -\frac{1}{3}\int \frac{dx}{-\frac{2}{3}+e^{-x}} = -\frac{1}{3}\left[-\frac{3x}{2} - \frac{3}{2}\ln\left|-\frac{2}{3}+e^{-x}\right|\right] + C = \\ &= \frac{x}{2} + \frac{1}{2}\ln\left|\frac{2-3e^{-x}}{-3}\right| + C = \frac{x}{2} + \frac{1}{2}\ln\left|2-3e^{-x}\right| + \frac{1}{2}\ln|-3| + C.\end{aligned}$$

Since the expression $\frac{1}{2}\ln|-3|$ is a constant, you can write

$$\int \frac{dx}{2-3e^{-x}} = \frac{x}{2} + \frac{1}{2}\ln\left|2-3e^{-x}\right| + C.$$

5. $\int (\ln x)^3 dx$
Solution. Aplly the reduction formula (see Appendix, formula 29)

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1}dx$$

to get

$$\begin{aligned}\int (\ln x)^3 dx &= x(\ln x)^3 - 3\int (\ln x)^2 dx = \\ &= x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\int (\ln x)dx\right) = \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6\left(x\ln x - \int dx\right) = \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C.\end{aligned}$$

Homework

In Problems 1 through 10, use one of the integration formulas listed in this section to find the given integral.

1.	$\int \frac{dx}{x(2x-3)}$	2.	$\int \frac{3dx}{4x(x-5)}$
3.	$\int \frac{dx}{\sqrt{x^2+25}}$	4.	$\int \frac{dx}{\sqrt{9x^2-4}}$
5.	$\int \frac{dx}{4-x^2}$	6.	$\int \frac{dx}{3x^2-9}$
7.	$\int \frac{dx}{3x^2+2x}$	8.	$\int \frac{4dx}{x^2-x}$
9.	$\int x^2e^{3x}dx$	10.	$\int x^3e^{-x}dx$

Locate a table of integrals and use it to find the integrals in Problems 11 through 16.

11.	$\int \frac{xdx}{2-x^2}$	12.	$\int \frac{x+3}{\sqrt{2x+4}}dx$
13.	$\int (\ln 2x)^2 dx$	14.	$\int \frac{dx}{3x\sqrt{2x+5}}$
15.	$\int \frac{xdx}{\sqrt{4-x^2}}$	16.	$\int \frac{dx}{\sqrt{3x^2-6x+2}}$

17. One table of integrals lists the formula

$$\int \frac{dx}{\sqrt{x^2 \pm p^2}} = \ln \left| \frac{x + \sqrt{x^2 \pm p^2}}{p} \right|$$

while another table lists

$$\int \frac{dx}{\sqrt{x^2 \pm p^2}} = \ln \left| x + \sqrt{x^2 \pm p^2} \right|.$$

Can you reconcile this apparent contradiction?

18. The following two formulas appear in a table of integrals:

$$\int \frac{dx}{p^2 - x^2} = \frac{1}{2p} \ln \left| \frac{p+x}{p-x} \right|$$

and

$$\int \frac{dx}{a + bx^2} = \frac{2}{2\sqrt{-ab}} \ln \left| \frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}} \right| \quad (\text{for} \quad -ab \geq 0).$$

(a) Use the second formula to derive the first.

(b) Apply both formulas to the integral $\int \frac{dx}{9-4x^2}$. Which do you find easier to use in this problem?

Results.

1.	$-\frac{1}{3} \ln \left\| \frac{x}{2x-3} \right\| + C$	2.	$-\frac{3}{20} \ln \left\| \frac{x}{x-5} \right\| + C$
3.	$\ln \left\| x + \sqrt{x^2 + 25} \right\| + C$	4.	$\frac{1}{3} \ln \left\| x + \sqrt{x^2 - \frac{4}{9}} \right\| + C$
5.	$\frac{1}{4} \ln \left(\frac{2+x}{2-x} \right) + C$	6.	$-\frac{1}{2\sqrt{3}} \ln \left\| \frac{\sqrt{3}+x}{\sqrt{3}-x} \right\| + C$
7.	$\frac{1}{2} \ln \left\| \frac{x}{3x+2} \right\| + C$	8.	$-4 \ln \left\| \frac{x}{x-1} \right\| + C$
9.	$\frac{1}{3} \left(x^2 - \frac{2}{3}x + \frac{2}{9} \right) e^{3x} + C$	10.	$-(x^3 + 3x^2 + 6x + 6)e^{-x} + C$
11.	$-\frac{1}{2} \ln \|2 - x^2\| + C$	12.	$\frac{1}{6} (2x+4) \sqrt{2x+4} + \sqrt{(2x+4)} + C$
13.	$x (\ln 2x)^2 - 2x \ln 2x + 2x + C$	14.	$\frac{1}{3\sqrt{5}} \ln \left\| \frac{\sqrt{2x+5}-\sqrt{5}}{\sqrt{2x+5}+\sqrt{5}} \right\| + C$
15.	$-\sqrt{(4-x^2)} + C$	16.	$\frac{1}{\sqrt{3}} \ln \left\| x - 1 + \sqrt{x^2 - 2x + \frac{2}{3}} \right\| + C$

17. $\ln \left(\frac{a}{b} \right) = \ln a - \ln b$

18. $\frac{1}{3} \ln \left| \frac{3+2x}{3-2x} \right| + C$

6.5 The Definite Integral

In problems 1 through 7 evaluate the given definite integral

1. $\int_{\ln\frac{1}{2}}^{2}\left(e^t - e^{-t}\right)dt$
   Solution.

$$\begin{aligned}\int_{\ln\frac{1}{2}}^{2}\left(e^t - e^{-t}\right)dt &= \left(e^t + e^{-t}\right)\Big|_{\ln\frac{1}{2}}^{2} = e^2 + e^{-2} - e^{\ln\frac{1}{2}} - e^{-\ln\frac{1}{2}} = \\ &= e^2 + e^{-2} - e^{\ln\frac{1}{2}} - e^{\ln 2} = e^2 + e^{-2} - \frac{1}{2} - 2 = \\ &= e^2 + e^{-2} - \frac{5}{2}.\end{aligned}$$

2. $\int_{-3}^{0}(2x+6)^4 dx$
   Solution. Substitute $u = 2x + 6$. Then $\frac{1}{2}du = dx$, $u(-3) = 0$, and $u(0) = 6$. Hence,

$$\int_{-3}^{0}(2x+6)^4 dx = \frac{1}{2}\int_0^6 u^4 du = \frac{1}{10}u^5\Big|_0^6 = \frac{6^5}{10} - 0 = \frac{3888}{5}.$$

3. $\int_1^2 \frac{x^2}{(x^3+1)^2}dx$
   Solution. Substitute $u = x^3 + 1$. Then $\frac{1}{3}du = x^2dx$, $u(1) = 2$, and $u(2) = 9$. Hence,

$$\int_1^2 \frac{x^2}{(x^3+1)^2}dx = \frac{1}{3}\int_2^9 u^{-2}du = -\frac{1}{3u}\Big|_2^9 = -\frac{1}{27} + \frac{1}{6} = \frac{7}{54}.$$

4. $\int_e^{e^2} \frac{1}{x\ln x}dx$
   Solution. Substitute $u = \ln x$. Then $du = \frac{1}{x}dx$, $u(e) = 1$, and $u(e^2) = 2$. Hence,

$$\int_e^{e^2} \frac{1}{x\ln x}dx = \int_1^2 \frac{du}{u} = \ln|u|\big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

5. $\int_{\frac{1}{2}}^{\frac{e}{2}} t\ln 2t dt$
   Solution. Since the factor t is easy to integrate and the factor $\ln 2t$ is simplified by differentiation, try integration by parts with

$$g(t) = t \qquad \text{and} \qquad f(t) = \ln 2t$$

Then,

$$G(t) = \int t dt = \frac{1}{2}t^2 \qquad \text{and} \qquad f'(t) = \frac{1}{t}$$

and so

$$\begin{aligned}
\int_{\frac{1}{2}}^{\frac{e}{2}} t \ln 2t dt &= \frac{1}{2}t^2 \ln 2t\Big|_{\frac{1}{2}}^{\frac{e}{2}} - \frac{1}{2}\int_{\frac{1}{2}}^{\frac{e}{2}} t dt = \frac{1}{2}t^2 \ln 2t - \frac{1}{4}t^2\Big|_{\frac{1}{2}}^{\frac{e}{2}} = \\
&= \frac{1}{8}e^2 \ln e - \frac{1}{16}e^2 - \frac{1}{8}\ln 1 + \frac{1}{16} = \frac{1}{8}e^2 - \frac{1}{16}e^2 + \frac{1}{16} = \\
&= \frac{1}{16}e^2 + \frac{1}{16} = \frac{1}{16}(e^2+1).
\end{aligned}$$

6. $\int_0^1 x^2 e^{2x} dx$
Solution. Apply the reduction formula $\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax} dx$ twice to get

$$\begin{aligned}
\int_0^1 x^2 e^{2x} dx &= \frac{1}{2}x^2 e^{2x}\Big|_0^1 - \int_0^1 x e^{2x} dx = \\
&= \frac{1}{2}x^2 e^{2x}\Big|_0^1 - \frac{1}{2}x e^{2x}\Big|_0^1 + \frac{1}{2}\int_0^1 e^{2x} dx = \\
&= \left(\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x}\right)\Big|_0^1 = \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x}\Big|_0^1 = \\
&= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4}\right)e^2 - \frac{1}{4}e^0 = \frac{1}{4}e^2 - \frac{1}{4} = \frac{1}{4}(e^2-1).
\end{aligned}$$

7. $\int_0^5 t e^{-\frac{5-t}{20}} dt$
Solution. Integrate by parts with

$$g(t) = e^{-\frac{5-t}{20}} \qquad \text{and} \qquad f(t) = t$$

Then,

$$G(t) = \int e^{-\frac{5-t}{20}} dt = 20e^{-\frac{5-t}{20}} \qquad \text{and} \qquad f'(t) = 1$$

and so

$$\begin{aligned}
\int_0^5 t e^{-\frac{5-t}{20}} dt &= 20te^{-\frac{5-t}{20}}\Big|_0^5 - 20\int_0^5 e^{-\frac{5-t}{20}} dt = \left(20te^{-\frac{5-t}{20}} - 400e^{-\frac{5-t}{20}}\right)\Big|_0^5 = \\
&= 20(t-20)e^{-\frac{5-t}{20}}\Big|_0^5 = 20\cdot(-15)e^0 - 20\cdot(-20)e^{-\frac{1}{4}} = \\
&= -300 + 400e^{-\frac{1}{4}} = 100\left(4e^{-\frac{1}{4}} - 3\right).
\end{aligned}$$

(a) Show that $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.

(b) Use the formula in part (a) to evaluate $\int_{-1}^1 |x|\, dx$.

(c) Evaluate $\int_0^4 (1+|x-3|)^2 dx$.

Solution. (a) By the Newton-Leibniz formula, you have

$$\begin{aligned}\int_a^b f(x)dx + \int_b^c f(x)dx &= F(b) - F(a) + F(c) - F(b) = \\ &= F(c) - F(a) = \int_a^c f(x)dx.\end{aligned}$$

(b) Since $|x| = -x$ for $x \le 0$ and $|x| = x$ for $x \ge 0$, you have to break the given integral into two integrals

$$\int_{-1}^0 |x|\, dx = \int_{-1}^0 (-x)dx = -\frac{1}{2}x^2\Big|_{-1}^0 = 0 + \frac{1}{2} = \frac{1}{2}$$

and

$$\int_0^1 |x|\, dx = \int_0^1 xdx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Thus,

$$\int_{-1}^1 |x|\, dx = \int_{-1}^0 |x|\, dx + \int_0^1 |x|\, dx = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) Since $|x-3| = -x+3$ for $x \le 3$ and $|x-3| = x-3$ for $x \ge 3$, you get

$$\begin{aligned}\int_0^4 (1+|x-3|)^2 dx &= \int_0^3 [1+(-x+3)]^2 dx + \int_3^4 [1+(x-3)]^2 dx = \\ &= \int_0^3 (-x+4)^2 dx + \int_3^4 (x-2)^2 dx = \\ &= -\frac{1}{3}(-x+4)^3\Big|_0^3 + \frac{1}{3}(x-2)^3\Big|_3^4 = \\ &= -\frac{1}{3} + \frac{64}{3} + \frac{8}{3} - \frac{1}{3} = \frac{70}{3}.\end{aligned}$$

9. (a) Show that if F is an antiderivative of f, then

$$\int_a^b f(-x)dx = -F(-b) + F(-a)$$

(b) A function f is said to be even if $f(-x) = f(x)$. [For example, $f(x) = x^2$ is even.] Use problem 8 and part (a) to show that if f is even, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

(c) Use part (b) to evaluate $\int_{-1}^{1} |x|\, dx$ and $\int_{-2}^{2} x^2 dx$.

(d) A function f is said to be odd if $f(-x) = -f(x)$. Use problem 8 and part (a) to show that if f is odd, then

$$\int_{-a}^{a} f(x)dx = 0.$$

(e) Evaluate $\int_{-12}^{12} x^3 dx$.

Solution. (a) Substitute $u = -x$. Then $du = -dx$, $u(a) = -a$ and $u(b) = -b$. Hence,

$$\int_{a}^{b} f(-x)dx = -\int_{-a}^{-b} f(u)du = -F(u)|_{-a}^{-b} = -F(-b) + F(-a).$$

(b) Since $f(-x) = f(x)$, you can write

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx.$$

By the part (a), you have

$$\begin{aligned}\int_{-a}^{0} f(-x)dx &= -F(0) + F(-(-a)) = F(a) - F(0) = \\ &= \int_{0}^{a} f(x)dx.\end{aligned}$$

Hence,

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

(c) Since $f(x) = |x|$ is an even function, you have

$$\int_{-1}^{1} |x|\, dx = 2\int_{0}^{1} |x|\, dx = 2\int_{0}^{1} xdx = x^2\Big|_{0}^{1} = 1 - 0 = 1.$$

Analogously,

$$\int_{-2}^{2} x^2 dx = 2\int_{0}^{2} x^2 dx = \frac{2}{3}x^3\Big|_{0}^{2} = \frac{16}{3}.$$

(d) Since $f(-x) = -f(x)$, you can write

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= -\int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx = \\ &= F(0) - F(a) + F(a) - F(0) = 0.\end{aligned}$$

(e) Since $f(x) = x^3$ is an odd function, you have

$$\int_{-12}^{12} x^3 dx = 0.$$

10. It is estimated that t days from now a farmer's crop will be increasing at the rate of $0.3t^2 + 0.6t + 1$ bushels per day. By how much will the value of the crop increase during the next 5 days if the market price remains fixed at 3 euros per bushel?
Solution. Let $Q(t)$ denote the farmer's crop t days from now. Then the rate of change of the crop with respect to time is

$$\frac{dQ}{dt} = 0.3t^2 + 0.6t + 1,$$

and the amount by which the crop will increase during the next 5 days is the definite integral

$$\begin{aligned}Q(5) - Q(0) &= \int_{0}^{5} \left(0.3t^2 + 0.6t + 1\right) dx = \left(0.1t^3 + 0.3t^2 + t\right)\Big|_0^5 = \\ &= 12.5 + 7.5 + 5 = 25 \text{ bushels}.\end{aligned}$$

Hence, the value of the market price will increase by $25 \cdot 3 = 75$ euros.

Homework

In problems 1 through 17, evaluate the given definite integral

1.	$\int_0^1 (x^4 - 3x^3 + 1)\,dx$	2.	$\int_{-1}^0 (3x^5 - 3x^2 + 2x - 1)\,dx$
3.	$\int_2^5 (2 + 2t + 3t^2)\,dt$	4.	$\int_1^9 \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) dt$
5.	$\int_1^3 \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) dx$	6.	$\int_{-3}^{-1} \frac{t+1}{t^3} dt$
7.	$\int_0^6 x^2(x-1)dx$	8.	$\int_1^2 (2x-4)^5 dx$
9.	$\int_0^4 \frac{1}{\sqrt{6t+1}} dt$	10.	$\int_0^1 (t^3 + t)\sqrt{t^4 + 2t^2 + 1}dt$
11.	$\int_0^1 \frac{6x}{x^2+1} dx$	12.	$\int_2^{e+1} \frac{x}{x-1} dx$
13.	$\int_1^2 (t+1)(t-2)^9 dt$	14.	$\int_1^{e^2} \ln t dt$
15.	$\int_{-2}^2 xe^{-x} dx$	16.	$\int_1^{e^2} \frac{(\ln x)^2}{x} dx$
17.	$\int_0^{10} (20+t)e^{-0.1t} dt$		

18. A study indicates that x months from now the population of a certain town will be increasing at the rate of $5+3x^{\frac{2}{3}}$ people per month. By how much will the population of the town increase over the next 8 months?

19. It is estimated that the demand for oil is increasing exponentially at the rate of 10 percent per year. If the demand for oil is currently 30 billion barrels per year, how much oil will be consumed during the next 10 years?

20. An object is moving so that its speed after t minutes is $5 + 2t + 3t^2$ meters per minute. How far does the object travel during the 2nd minute?

Results.

1.	$\frac{9}{20}$	2.	$-\frac{7}{2}$	3.	144	4.	$\frac{40}{3}$
5.	$\frac{8}{3} + \ln 3$	6.	$\frac{2}{9}$	7.	252	8.	$-\frac{16}{3}$
9.	$\frac{4}{3}$	10.	$\frac{7}{6}$	11.	$3 \ln 2$	12.	e
13.	$-\frac{23}{110}$	14.	$e^2 + 1$	15.	$-3e^{-2} - e^2$	16.	$\frac{8}{3}$
17.	152.85	18.	98 people	19.	515.48 billion barrels	20.	15 meters

6.6 Area and Integration

In problems 1 through 9 find the area of the region R.

1. R is the triangle with vertices $(-4, 0)$, $(2, 0)$ and $(2, 6)$.
Solution. From the corresponding graph (Figure 6.1) you see that the region in question is bellow the line $y = x + 4$ above the x axis, and extends from $x = -4$ to $x = 2$.

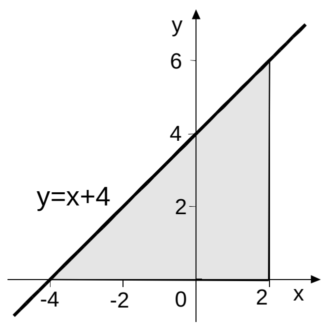

Figure 6.1.

Hence,

$$A = \int_{-4}^{2} (x + 4)dx = \left(\frac{1}{2}x^2 + 4x\right)\Big|_{-4}^{2} = (2 + 8) - (8 - 16) = 18.$$

2. R is the region bounded by the curve $y = e^x$, the lines $x = 0$ and $x = \ln\frac{1}{2}$, and the x axis.
Solution. Since $\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2 \simeq -0.7$, from the corresponding graph (Figure 6.2) you see that the region in question is bellow the line $y = e^x$ above the x axis, and extends from $x = \ln\frac{1}{2}$ to $x = 0$.

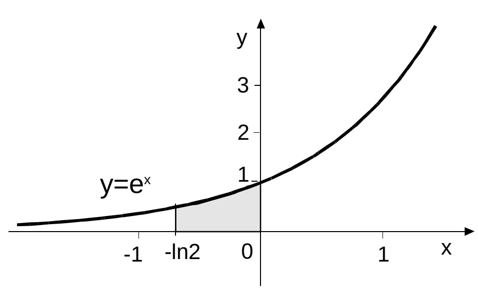

Figure 6.2.

Hence,

$$A = \int_{\ln \frac{1}{2}}^{0} e^x dx = e^x\Big|_{\ln \frac{1}{2}}^{0} = e^0 - e^{\ln \frac{1}{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

3. R is the region in the first quadrant that lies below the curve $y = x^2 + 4$ and is bounded by this curve, the line $y = -x + 10$, and the coordinate axis.
Solution. First sketch the region as shown in Figure 6.3. Note that the curve $y = x^2 + 4$ and the line $y = -x + 10$ intersect in the first quadrant at the point $(2, 8)$, since $x = 2$ is the only positive solution of the equation $x^2 + 4 = -x + 10$, i.e. $x^2 + x - 6 = 0$. Also note that the line $y = -x + 10$ intersects the x axis at the point $(10, 0)$.

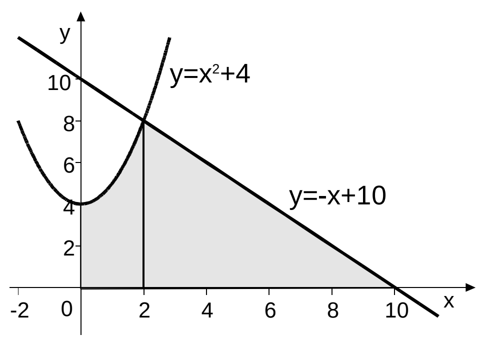

Figure 6.3.

Observe that to the left of $x = 2$, R is bounded above by the curve $y = x^2 + 4$, while to the right of $x = 2$, it is bounded by the line $y = -x + 10$. This suggests that you break R into two subregions, R_1 and R_2, as shown in Figure 6.3, and apply the integral formula for area to each subregion separately. In particular,

$$A_1 = \int_0^2 (x^2 + 4)dx = \left(\frac{1}{3}x^3 + 4x\right)\Big|_0^2 = \frac{8}{3} + 8 = \frac{32}{3}$$

and

$$A_2 = \int_2^{10} (-x + 10)dx = \left(-\frac{1}{2}x^2 + 10x\right)\Big|_2^{10} = -50 + 100 + 2 - 20 = 32.$$

Therefore,

$$A = A_1 + A_2 = \frac{32}{3} + 32 = \frac{128}{3}.$$

4. R is the region bounded by the curves $y = x^2 + 5$ and $y = -x^2$, the line $x = 3$, and the y axis.
Solution. Sketch the region as shown in Figure 6.4.

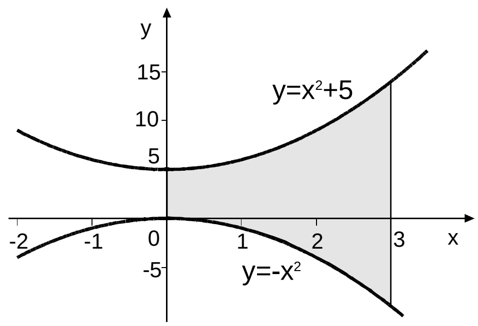

Figure 6.4.

Notice that the region in question is bounded above by the curve $y = x^2 + 5$ and below by the curve $y = -x^2$ and extends from $x = 0$ to $x = 3$. Hence,

$$A = \int_0^3 [(x^2+5)-(-x^2)]dx = \int_0^3 (2x^2+5)dx = \left(\frac{2}{3}x^3 + 5x\right)\Big|_0^3 = 18+15 = 33.$$

5. R is the region bounded by the curves $y = x^2 - 2x$ and $y = -x^2 + 4$.
Solution. First make a sketch of the region as shown in Figure 6.5 and find the points of intersection of the two curves by solving the equation

$$x^2 - 2x = -x^2 + 4 \quad \text{i.e.} \quad 2x^2 - 2x - 4 = 0$$

to get

$$x = -1 \quad \text{and} \quad x = 2.$$

The corresponding points $(-1, 3)$ and $(2, 0)$ are the points of intersection.

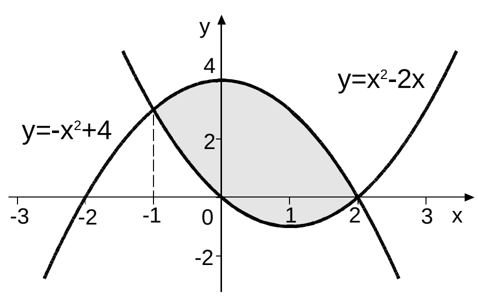

Figure 6.5.

Notice that for $-1 \le x \le 2$, the graph of $y = -x^2 + 4$ lies above that of $y = x^2 - 2x$. Hence,

$$\begin{aligned} A &= \int_{-1}^{2} [(-x^2+4) - (x^2-2x)]dx = \int_{-1}^{2} (-2x^2+2x+4)dx = \\ &= \left(-\frac{2}{3}x^3 + x^2 + 4x\right)\Big|_{-1}^{2} = -\frac{16}{3} + 4 + 8 - \frac{2}{3} - 1 + 4 = 9. \end{aligned}$$

6. R is the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.
Solution. Sketch the region as shown in Figure 6.6. Find the points of intersection by solving the equations of the two curves simultaneously to get

$$x^2 = \sqrt{x} \qquad x^2 - \sqrt{x} = 0 \qquad \sqrt{x}(x^{\frac{3}{2}} - 1) = 0$$

$$x = 0 \qquad \text{and} \qquad x = 1.$$

The corresponding points $(0,0)$ and $(1,1)$ are the points of intersection.

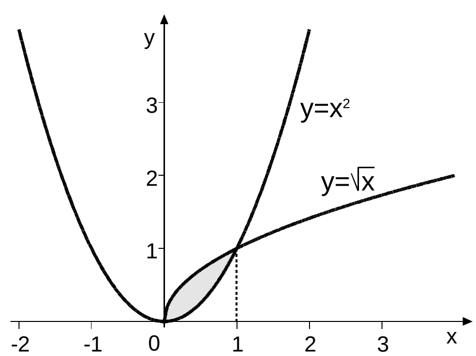

Figure 6.6.

Notice that for $0 \le x \le 1$, the graph of $y = \sqrt{x}$ lies above that of $y = x^2$. Hence,

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

(a) R is the region to the right of the y axis that is bounded above by the curve $y = 4 - x^2$ and below the line $y = 3$.

(b) R is the region to the right of the y axis that lies below the line $y = 3$ and is bounded by the curve $y = 4 - x^2$, the line $y = 3$, and the coordinate axes.

Solution. Note that the curve $y = 4 - x^2$ and the line $y = 3$ intersect to the right of the y axis at the point $(1, 3)$, since $x = 1$ is the positive solution of the equation $4 - x^2 = 3$, i.e. $x^2 = 1$.
(a) Sketch the region as shown in Figure 6.7.

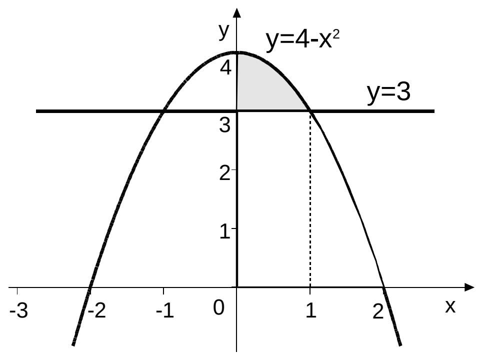

Figure 6.7.

Notice that for $0 \leq x \leq 1$, the graph of $y = 4 - x^2$ lies above that of $y = 3$. Hence,

$$A = \int_0^1 (4-x^2-3)dx = \int_0^1 (1-x^2)dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^1 = 1-\frac{1}{3} = \frac{2}{3}.$$

(b) Sketch the region as shown in Figure 6.8.

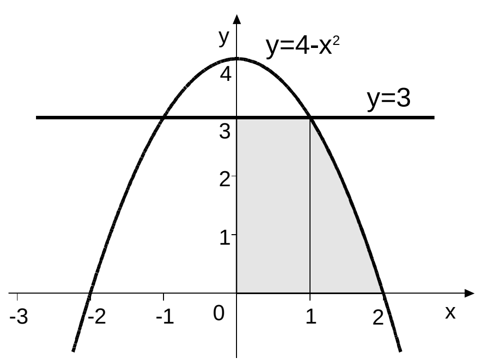

Figure 6.8.

Observe that to the left of $x = 1$, R is bounded above by the curve $y = 3$, while to the right of $x = 1$, it is bounded by the line $y = 4 - x^2$. This suggests that you break R into two subregions, R_1 and R_2, as shown in Figure 6.8, and apply the integral formula for area to each subregion separately. In particular,

$$A_1 = \int_0^1 3dx = 3x\Big|_0^1 = 3$$

and

$$A_2 = \int_1^2 (4 - x^2)dx = \left(4x - \frac{1}{3}x^3\right)\Bigg|_1^2 = 8 - \frac{8}{3} - 4 + \frac{1}{3} = \frac{5}{3},$$

so

$$A = A_1 + A_2 = 3 + \frac{5}{3} = \frac{14}{3}.$$

7. R is the region bounded by the curve $y = \frac{1}{x^2}$ and the lines $y = x$ and $y = \frac{x}{8}$.
Solution. First make a sketch of the region as shown in Figure 6.9 and find the points of intersection of the curve and the lines by solving the equations

$$\frac{1}{x^2} = x \quad \text{and} \quad \frac{1}{x^2} = \frac{x}{8} \quad \text{i.e.} \quad x^3 = 1 \quad \text{and} \quad x^3 = 8$$

to get

$$x = 1 \quad \text{and} \quad x = 2.$$

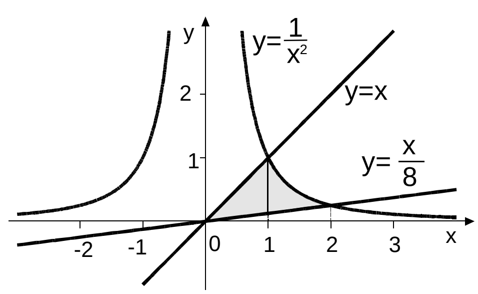

Figure 6.9.

Then break R into two subregions, R_1 that extends from $x = 0$ to $x = 1$ and R_2 that extends from $x = 1$ to $x = 2$, as shown in Figure 6.9. Hence, the area of the region R_1 is

$$A_1 = \int_0^1 \left(x - \frac{x}{8}\right) dx = \int_0^1 \frac{7}{8}xdx = \frac{7}{16}x^2\Bigg|_0^1 = \frac{7}{16}$$

and the area of the region R_2 is

$$A_2 = \int_1^2 \left(\frac{1}{x^2} - \frac{x}{8}\right) dx = \left(-\frac{1}{x} - \frac{1}{16}x^2\right)\Big|_1^2 = -\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{16} = \frac{5}{16}.$$

Thus, the area of the region R is the sum

$$A = A_1 + A_2 = \frac{12}{16} = \frac{3}{4}.$$

8. R is the region bounded by the curves $y = x^3 - 2x^2 + 5$ and $y = x^2 + 4x - 7$.

Solution. First make a rough sketch of the two curves as shown in Figure 6.10. You find the points of intersection by solving the equations of the two curves simultaneously

$$\begin{aligned} x^3 - 2x^2 + 5 &= x^2 + 4x - 7 \qquad x^3 - 3x^2 - 4x + 12 = 0 \\ x^2(x-3) - 4(x-3) &= 0 \qquad\qquad (x-3)(x-2)(x+2) = 0 \end{aligned}$$

to get

$$x = -2, \qquad x = 2 \qquad \text{and} \qquad x = 3.$$

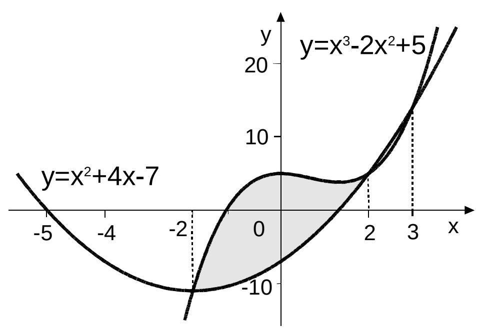

Figure 6.10.

The region whose area you wish to compute lies between $x = -2$ and $x = 3$, but since the two curves cross at $x = 2$, neither curve is always above the other between $x = -2$ and $x = 3$. However, since the curve $y = x^3 - 2x^2 + 5$ is above $y = x^2 + 4x - 7$ between $x = -2$ and $x = 2$, and since $y = x^2 + 4x - 7$ is above $y = x^3 - 2x^2 + 5$ between $x = 2$ and $x = 3$, it follows that the area of the region between $x = -2$ and $x = 2$, is

$$\begin{aligned} A_1 &= \int_{-2}^{2} \left[\left(x^3 - 2x^2 + 5\right) - \left(x^2 + 4x - 7\right)\right] dx = \\ &= \int_{-2}^{2} \left(x^3 - 3x^2 - 4x + 12\right) dx = \\ &= \left(\frac{1}{4}x^4 - x^3 - 2x^2 + 12x\right)\Big|_{-2}^{2} = \\ &= 4 - 8 - 8 + 24 - 4 - 8 + 8 + 24 = 32 \end{aligned}$$

and the area of the region between $x = 2$ and $x = 3$, is

$$\begin{aligned} A_2 &= \int_{2}^{3} \left[\left(x^2 + 4x - 7\right) - \left(x^3 - 2x^2 + 5\right)\right] dx = \\ &= \int_{2}^{3} \left(-x^3 + 3x^2 + 4x - 12\right) dx = \\ &= \left(-\frac{1}{4}x^4 + x^3 + 2x^2 - 12x\right)\Big|_{2}^{3} = \\ &= -\frac{81}{4} + 27 + 18 - 36 + 4 - 8 - 8 + 24 = \\ &= -\frac{81}{4} + 21 = \frac{3}{4}. \end{aligned}$$

Thus, the total area is the sum

$$A = A_1 + A_2 = 32 + \frac{3}{4} = \frac{131}{4}.$$

Homework

In problems 1 through 20 find the area of the region R.

1. R is the triangle bounded by the line $y = 4 - 3x$ and the coordinate axes.
2. R is the rectangle with vertices $(1, 0)$, $(-2, 0)$, $(-2, 5)$ and $(1, 5)$.
3. R is the trapezoid bounded by the lines $y = x + 6$ and $x = 2$ and the coordinate axes.
4. R is the region bounded by the curve $y = \sqrt{x}$, the line $x = 4$, and the x axis.

5. R is the region bounded by the curve $y = 4x^3$, the line $x = 2$, and the x axis.

6. R is the region bounded by the curve $y = 1 - x^2$ and the x axis.

7. R is the region bounded by the curve $y = -x^2 - 6x - 5$ and the x axis.

8. R is the region in the first quadrant bounded by the curve $y = 4 - x^2$ and the lines $y = 3x$ and $y = 0$.

9. R is the region bounded by the curve $y = \sqrt{x}$ and the lines $y = 2 - x$ and $y = 0$.

10. R is the region in the first quadrant that lies under the curve $y = \frac{16}{x}$ and that is bounded by this curve and the lines $y = x$, $y = 0$, and $x = 8$.

11. R is the region bounded by the curve $y = x^2 - 2x$ and the x axis. (Hint: Reflect the region across the x axis and integrate the corresponding function.)

12. R is the region bounded by the curves $y = x^2 + 3$ and $y = 1 - x^2$ between $x = -2$ and $x = 1$.

13. R is the region bounded by the curve $y = e^x$ and the lines $y = 1$ and $x = 1$.

14. R is the region bounded by the curve $y = x^2$ and the line $y = x$.

15. R is the region bounded by the curve $y = x^2$ and the line $y = 4$.

16. R is the region bounded by the curves $y = x^3 - 6x^2$ and $y = -x^2$.

17. R is the region bounded by the line $y = x$ and the curve $y = x^3$.

18. R is the region in the first quadrant bounded by the curve $y = x^2 + 2$ and the lines $y = 11 - 8x$ and $y = 11$.

19. R is the region bounded by the curves $y = x^2 - 3x + 1$ and $y = -x^2 + 2x + 2$.

20. R is the region bounded by the curves $y = x^3 - x$ and $y = -x^2 + x$.

Results.

1. $\frac{8}{3}$	2. 15	3. 14	4. $\frac{16}{3}$	5. 16
6. $\frac{4}{3}$	7. $\frac{32}{3}$	8. $\frac{19}{6}$	9. $\frac{7}{6}$	10. $8(1 + \ln 4)$
11. $\frac{4}{3}$	12. 12	13. $e - 2$	14. $\frac{1}{6}$	15. $\frac{32}{3}$
16. $\frac{625}{12}$	17. $\frac{1}{2}$	18. $\frac{40}{3}$	19. $\frac{11}{8}\sqrt{33}$	20. $\frac{37}{12}$